360VO: Visual Odometry Using A Single 360 Camera
(Deduction of Jacobian)

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7. Spherical Camera Model

$$x_c^2 + y_c^2 + z_c^2 = 1$$

2D→3D:

$$\text{longitude} = \left(\frac{u}{W} - \frac{1}{2}\right)\cdot 2\pi$$
$$\text{latitude} = \left(\frac{1}{2} - \frac{v}{H}\right)\cdot \pi$$
$$x_c = \cos(lat)\sin(lon)$$
$$y_c = -\sin(lat)$$
$$z_c = \cos(lat)\cos(lon)$$

3D→2D:

$$X_c = (R_{cw} P_w + t_{cw}).\text{normalized}()$$

$$\text{latitude} = -\arcsin(y_c)$$
$$\text{longitude} = \arctan\left(\frac{x_c}{z_c}\right)$$
$$u = W\cdot\left(0.5 + \frac{\text{longitude}}{2\pi}\right)$$
$$v = H\cdot\left(0.5 - \frac{\text{latitude}}{\pi}\right)$$

Image $X = (u, v)$ | Spherical Image $S = (lon, lat)$ | Camera $X_c = (X_c, Y_c, Z_c)$

$$\begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \overset{K}{\begin{bmatrix} f_x & & c_x \\ & f_y & c_y \\ & & 1 \end{bmatrix}} \begin{bmatrix} lon \\ lat \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{W}{2\pi} & & \frac{W}{2} \\ & -\frac{H}{\pi} & \frac{H}{2} \\ & & 1 \end{bmatrix} \begin{bmatrix} lon \\ lat \\ 1 \end{bmatrix}$$

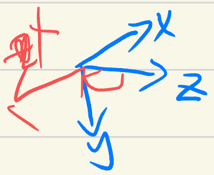
$$\begin{cases} \arctan \frac{\bar{X}_c}{\bar{Z}_c} \\ -\arcsin \bar{Y}_c \end{cases}$$

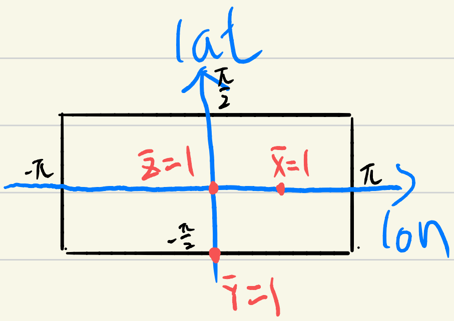
$$\rho = \frac{1}{\sqrt{X_c^2 + Y_c^2 + Z_c^2}}$$ inverse depth

$$\bar{X}_c = \rho X_c$$

$$\begin{bmatrix} lon \\ lat \\ 1 \end{bmatrix} = K^{-1} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} f_x^{-1} & 0 & -f_x^{-1}c_x \\ & f_y^{-1} & -f_y^{-1}c_y \\ & & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$
$$= \begin{bmatrix} \frac{2\pi}{W} & 0 & -\pi \\ & -\frac{\pi}{H} & \frac{\pi}{2} \\ & & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \bar{X}_c \\ \bar{Y}_c \\ \bar{Z}_c \end{bmatrix} = \begin{bmatrix} \cos(lat)\cdot\sin(lon) \\ -\sin(lat) \\ \cos(lat)\cdot\cos(lon) \end{bmatrix}$$





given matched points $x_1 = \begin{bmatrix} u_1 \\ v_1 \\ 1 \end{bmatrix}$ $x_2 = \begin{bmatrix} u_2 \\ v_2 \\ 1 \end{bmatrix}$ relative R, t

1. project to unit sphere, $\bar{X}_{c1}$ $\bar{X}_{12}$

$$\rho_1 / \rho_2 \, \bar{X}_{12} = R\bar{X}_{c1} + t\rho_1$$

$$\frac{\bar{X}_{12}}{\bar{Z}_{c2}} = \frac{R\bar{X}_{c1}[0] + t[0]\rho_1}{R\bar{X}_{c1}[2] + t[2]\rho_1}$$

$$\rho_1 = \frac{\frac{\bar{X}_{12}}{\bar{Z}_{12}} R\bar{X}_{c1}[2] - R\bar{X}_{c1}[0]}{t[0] - \frac{\bar{X}_{12}}{\bar{Z}_{12}} t[2]}$$

8个参数：相对位姿 $se(3)$，后两个是光度仿射变换

Ref → new 8 parameters : relative pose $se(3)$, affine photometric transform

$$I_2 = a_{21} I_1 + b_{21}$$

$$a_{21} = \frac{e^{a_2}\Delta t_2}{e^{a_1}\Delta t_1} \quad b_{21} = b_2 - a_{21} b_1$$

Initially, $a_1, b_1 = [0,0]$, so

$$a_{21} = \frac{e^{a_2}\Delta t_2}{\Delta t_1} \quad b_{21} = b_2$$

photometric error $\quad r = w(I_2[x_2] - (a_{21} I_1[x_1] + b_{21}))$

$x_2 = f(x_1, \xi_{21}, \rho_1)$ 两帧相对位姿，x_1 在1中的逆深度

6、 $\frac{\partial r_{21}}{\partial a_{21}} = -w_h I_1[x_1]$

$$\frac{\partial r_{21}}{\partial a_2} = \frac{\partial r_{21}}{\partial a_{21}} \cdot \frac{\partial a_{21}}{\partial a_2} = -w_h a_{21} I_1[x_1]$$

7、 $\frac{\partial r_{21}}{\partial b_{21}} = -w_h = \frac{\partial r_{21}}{\partial b_2}$

- $\frac{\partial r_{21}}{\partial x_2} = [g_{u_2}, g_{v_2}, 0]$

- $$\frac{\partial x_2}{\partial S_2} = \begin{bmatrix} \frac{\partial u_2}{\partial lon_2} & \frac{\partial u_2}{\partial lat_2} & \frac{\partial u_2}{\partial 1} \\ \frac{\partial v_2}{\partial lon_2} & \frac{\partial v_2}{\partial lat_2} & \frac{\partial v_2}{\partial 1} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} f_x & & \\ & f_y & \\ & & 0 \end{bmatrix}$$

- $$\frac{\partial S_2}{\partial \bar{x}_{c2}} = \begin{bmatrix} \frac{\partial lon_2}{\partial \bar{x}_{c2}} & \frac{\partial lon_2}{\partial \bar{y}_{c2}} & \frac{\partial lon_2}{\partial \bar{z}_{c2}} \\ \frac{\partial lat_2}{\partial \bar{x}_{c2}} & \frac{\partial lat_2}{\partial \bar{y}_{c2}} & \frac{\partial lat_2}{\partial \bar{z}_{c2}} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2} & 0 & -\frac{\bar{x}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2} \\ & \frac{1}{\sqrt{1-\bar{y}_{c2}^2}} & \\ & & 0 \end{bmatrix}$$

- $$\frac{\partial \bar{x}_{c2}}{\partial \rho_1} = \frac{\partial [\rho_2 (R_{21}\rho_1^{-1}\bar{x}_{c1} + t_{21})]}{\partial \rho_1} = \frac{\partial \rho_2}{\partial \rho_1}(X_{c2}) + \rho_2 \frac{\partial (R_{21}\rho_1^{-1}\bar{x}_{c1} + t_{21})}{\partial \rho_1}$$

$$\rho_2 = \frac{1}{\|X_{c2}\|} = \frac{1}{\|\rho_1^{-1} R_{21}\bar{x}_{c1} + t_{21}\|} = \frac{1}{\sqrt{(\rho_1^{-1} r_{21}^1 \bar{x}_{c1} + t_{21}^x)^2 + (\rho_1^{-1} r_{21}^2 \bar{x}_{c1} + t_{21}^y)^2 + (\rho_1^{-1} r_{21}^3 \bar{x}_{c1} + t_{21}^z)^2}}$$

$$= \rho_2^3 \rho_1^{-2} [a_x(a_x\rho_1^{-1} + t_{21}^x) + a_y(a_y\rho_1^{-1} + t_{21}^y) + a_z(a_z\rho_1^{-1} + t_{21}^z)] \cdot X_{c2}$$

$$- \rho_2 \rho_1^{-2} R_{21}\bar{X}_{c1}$$

$$= \rho_2^3 \rho_1^{-2} [a_x \ a_y \ a_z] X_{c2} \cdot X_{c2} - \rho_2 \rho_1^{-2} R_{21} \bar{X}_{c1}$$

$$= \rho_2 \rho_1^{-2} ([a_x \ a_y \ a_z] \bar{X}_{c2} \cdot \bar{X}_{c2} - [a_x \ a_y \ a_z]^T)$$

$$= \rho_2 \rho_1^{-1} ([A_x \ A_y \ A_z] \bar{X}_{c2} \cdot \bar{X}_{c2} - [A_x \ A_y \ A_z]^T)$$

$$\frac{\partial \bar{X}_{c2}}{\partial \xi_{21}} = \frac{\partial \rho_2 X_{c2}}{\partial \varepsilon_{21}} = \frac{\partial \rho_2}{\partial \varepsilon_{21}} X_{21} + \rho_2 \frac{\partial X_{c2}}{\partial \varepsilon_{21}} \qquad \rho_2 = \frac{1}{\sqrt{X_c^2 + Y_c^2 + Z_c^2}}$$

$$\begin{aligned}
\frac{\partial \rho_2}{\partial \varepsilon_{21}} &= \frac{\partial \rho_2}{\partial X_{c2}} \cdot \frac{\partial X_{c2}}{\partial \varepsilon_{21}} + \frac{\partial \rho_2}{\partial Y_{c2}} \cdot \frac{\partial Y_{c2}}{\partial \varepsilon_{21}} + \frac{\partial \rho_2}{\partial Z_{c2}} \cdot \frac{\partial Z_{c2}}{\partial \varepsilon_{21}} \\
&= -X_{c2}\rho_2^3 [1\ 0\ 0\ 0\ Z_{c2}\ -Y_{c2}] - Y_{c2}\rho_2^3 [0\ 1\ 0\ -Z_{c2}\ 0\ X_{c2}] \\
&\quad - Z_{c2}\rho_2^3 [0\ 0\ 1\ Y_{c2}\ -X_{c2}\ 0] \\
&= -\rho_2^3 [X_{c2}\ Y_{c2}\ Z_{c2}\ 0\ 0\ 0]
\end{aligned}$$

$$\frac{\partial X_{c2}}{\partial \varepsilon_{21}} = \begin{bmatrix} \frac{\partial X_{c2}}{\partial \varepsilon_{21}} \\ \frac{\partial Y_{c2}}{\partial \varepsilon_{21}} \\ \frac{\partial Z_{c2}}{\partial \varepsilon_{21}} \end{bmatrix} = \begin{bmatrix} 1 & & & 0 & Z_{c2} & -Y_{c2} \\ & 1 & & -Z_{c2} & 0 & X_{c2} \\ & & 1 & Y_{c2} & -X_{c2} & 0 \end{bmatrix}$$

$$\frac{\partial \bar{X}_{c2}}{\partial \xi_{21}} = -\rho_2^3 \begin{bmatrix} X_{c2}^2 & Y_{c2}X_{c2} & Z_{c2}X_{c2} \\ X_{c2}Y_{c2} & Y_{c2}^2 & Z_{c2}Y_{c2} & 0 \\ X_{c2}Z_{c2} & Y_{c2}Z_{c2} & Z_{c2}^2 \end{bmatrix} + \rho_2 \begin{bmatrix} 1 & & & 0 & Z_{c2} & -Y_{c2} \\ & 1 & & -Z_{c2} & 0 & X_{c2} \\ & & 1 & Y_{c2} & -X_{c2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \rho_2 - \rho_2 \bar{X}_{c2}^2 & -\rho_2 \bar{Y}_{c2}\bar{X}_{c2} & -\rho_2 \bar{Z}_{c2}\bar{X}_{c2} & 0 & \bar{Z}_{c2} & -\bar{Y}_{c2} \\ -\rho_2 \bar{X}_{c2}\bar{Y}_{c2} & \rho_2 - \rho_2 \bar{Y}_{c2}^2 & -\rho_2 \bar{Z}_{c2}\bar{Y}_{c2} & -\bar{Z}_{c2} & 0 & \bar{X}_{c2} \\ -\rho_2 \bar{X}_{c2}\bar{Z}_{c2} & -\rho_2 \bar{Y}_{c2}\bar{Z}_{c2} & \rho_2 - \rho_2 \bar{Z}_{c2}^2 & \bar{Y}_{c2} & -\bar{X}_{c2} & 0 \end{bmatrix}$$

- $$\frac{\partial X_2}{\partial \varepsilon_{21}} = \begin{bmatrix} f_x & & \\ & f_y & 0 \end{bmatrix} \begin{bmatrix} \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} & 0 & -\frac{\bar{X}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \\ & \frac{1}{\sqrt{1-\bar{Y}^2}} & \\ & & 0 \end{bmatrix} \cdot \frac{\partial \bar{X}_{c2}}{\partial \xi_{21}}$$

$$= \begin{bmatrix} f_x \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} & 0 & -f_x \frac{\bar{X}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \\ 0 & -\frac{f_y}{\sqrt{1-\bar{Y}^2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \rho_2 - \rho_2 \bar{X}_{c2}^2 & -\rho_2 \bar{X}_{c2}\bar{Y}_{c2} & -\rho_2 \bar{X}_{c2}\bar{Z}_{c2} & 0 & \bar{Z}_{c2} & -\bar{Y}_{c2} \\ -\rho_2 \bar{X}_{c2}\bar{Y}_{c2} & \rho_2 - \rho_2 \bar{Y}_{c2}^2 & -\rho_2 \bar{Z}_{c2}\bar{Y}_{c2} & -\bar{Z}_{c2} & 0 & \bar{X}_{c2} \\ -\rho_2 \bar{X}_{c2}\bar{Z}_{c2} & -\rho_2 \bar{Y}_{c2}\bar{Z}_{c2} & \rho_2 - \rho_2 \bar{Z}_{c2}^2 & \bar{Y}_{c2} & -\bar{X}_{c2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} f_x \rho_2 \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2}, & 0, & -f_x \rho_2 \frac{\bar{X}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2}, & -f_x \frac{\bar{X}_{c2}\bar{Y}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2}, & f_x, & -f_x \frac{\bar{Y}_{c2}\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \\ f_y \rho_2 \frac{\bar{X}_{c2}\bar{Y}_{c2}}{\sqrt{1-\bar{Y}^2}}, & -f_y \rho_2 \sqrt{1-\bar{Y}_{c2}^2}, & f_y \rho_2 \frac{\bar{Z}_{c2}\bar{Y}_{c2}}{\sqrt{1-\bar{Y}_{c2}^2}}, & f_y \frac{\bar{Z}_{c2}}{\sqrt{1-\bar{Y}_{c2}^2}}, & 0 & -f_y \frac{\bar{X}_{c2}}{\sqrt{1-\bar{Y}_{c2}^2}} \end{bmatrix}$$

- $$\frac{\partial X_2}{\partial \rho_1} = \begin{bmatrix} f_x \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} & 0 & -f_x \frac{\bar{X}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \\ 0 & -\frac{f_y}{\sqrt{1-\bar{Y}^2}} & 0 \\ 0 & 0 & 0 \end{bmatrix} \rho_2 \rho_1^{-2} \left([a_x\ a_y\ a_z] \cancel{X}_{c2} \cdot \cancel{X}_{c2} - [a_x\ a_y\ a_z]^T \right)$$

$\rho_2 \rho_1^{-1} ([A_x\ A_y\ A_z] \bar{X}_{c2} \cdot \bar{X}_{c2} - [A_x\ A_y\ A_z]^T)$ // 3X3

$$= \begin{bmatrix} f_x \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \cdot \rho_2 \rho_1^{-2} ([a_x\ a_y\ a_z] \cancel{X}_{c2} \cdot \bar{X}_{c2} - a_x) - f_x \frac{\bar{X}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \rho_2 \rho_1^{-2} ([a_x\ a_y\ a_z] \cancel{X}_{c2} \cdot \bar{Z}_{c2} - a_x) \\ -\frac{f_y}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \rho_2 \rho_1^{-2} ([a_x\ a_y\ a_z] \cancel{X}_{c2} \cdot \bar{Y}_{c2} - a_y) \end{bmatrix}$$

$$= \begin{bmatrix} f_x \frac{\bar{Z}_{c2}}{\bar{Z}_{c2}^2 + \bar{X}_{c2}^2} \cdot \rho_2 \rho_1^{-2} [(a_x \bar{X}_{c2} + a_y \bar{Y}_{c2} + a_z \bar{Z}_{c2}) \cdot \bar{X}_{c2} - a_x] \sim \\ -\frac{f_y}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \rho_2 \rho_1^{-2} [(a_x \bar{X}_{c2} + a_y \bar{Y}_{c2} + a_z \bar{Z}_{c2}) \cdot \bar{Y}_{c2} - a_Y] \end{bmatrix}$$

$$\frac{\partial X_2}{\partial K} = \begin{bmatrix} \frac{\partial u_2}{\partial f_x} & \frac{\partial u_2}{\partial f_y} & \frac{\partial u_2}{\partial c_x} & \frac{\partial u_2}{\partial c_y} \\ \frac{\partial v_2}{\partial f_x} & \frac{\partial v_2}{\partial f_y} & \frac{\partial v_2}{\partial c_x} & \frac{\partial v_2}{\partial c_y} \end{bmatrix}$$

$$\frac{\partial u_2}{\partial f_x} = lon_2 + f_x \frac{\partial lon_2}{\partial f_x} \qquad \frac{\partial u_2}{\partial f_y} = f_x \frac{\partial lon_2}{\partial f_y}$$

$$\frac{\partial u_2}{\partial c_x} = f_x \frac{\partial lon_2}{\partial c_x} + 1 \qquad \frac{\partial u_2}{\partial c_y} = f_x \frac{\partial lon_2}{\partial c_y}$$

$$\frac{\partial v_2}{\partial f_x} = f_y \frac{\partial lat_2}{\partial f_x} \qquad \frac{\partial v_2}{\partial f_y} = lat_2 + f_y \frac{\partial lat_2}{\partial f_y}$$

$$\frac{\partial v_2}{\partial c_x} = f_y \frac{\partial lat_2}{\partial c_x} \qquad \frac{\partial v_2}{\partial c_y} = f_y \frac{\partial lat_2}{\partial c_y} + 1$$

$$\bar{X}_{c2} = \rho_2 (R_{21} \rho_1^{-1} \bar{X}_{c1} + t_{21})$$

$$= \rho_2 \rho_1^{-1} \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \begin{bmatrix} \cos[f_y^{-1}(v_1 - c_y)] \sin[f_x^{-1}(u_1 - c_x)] \\ -\sin[f_y^{-1}(v_1 - c_y)] \\ \cos[f_y^{-1}(v_1 - c_y)] \cos[f_x^{-1}(u_1 - c_x)] \end{bmatrix} + \rho_2 \begin{bmatrix} t_{21}^x \\ t_{21}^y \\ t_{21}^z \end{bmatrix}$$

$$= \rho_2 \rho_1^{-1} \begin{bmatrix} r_{11} F(1) + r_{12} F(2) + r_{13} F(3) \\ r_{21} F(1) + r_{22} F(2) + r_{23} F(3) \\ r_{31} F(1) + r_{32} F(2) + r_{33} F(3) \end{bmatrix} + \rho_2 \begin{bmatrix} t_{21}^x \\ t_{21}^y \\ t_{21}^z \end{bmatrix} = \begin{bmatrix} \bar{X}_{c2} \\ \bar{Y}_{c2} \\ \bar{Z}_{c2} \end{bmatrix}$$

$$lon_2 = \arctan \frac{\bar{X}_{c2}}{\bar{Z}_{c2}} = \arctan \frac{\rho_2 \rho_1^{-1} (r_{11} F(1) + r_{12} F(2) + r_{13} F(3)) + t_{21}^x \rho_2}{\rho_2 \rho_1^{-1} (r_{31} F(1) + r_{32} F(2) + r_{33} F(3)) + t_{21}^z \cdot \rho_2}$$

$$lat_2 = -\arcsin \bar{Y}_{c2} = -\arcsin [\rho_2 \rho_1^{-1} (r_{21} F(1) + r_{22} F(2) + r_{23} F(3)) + \rho_2 t_{21}^y)]$$

$$\frac{\partial lon_2}{\partial f_x} = \frac{\partial \arctan \frac{\bar{x}_{c2}}{\bar{z}_{c2}}}{\partial f_x} = \frac{1}{1+(\frac{\bar{x}_{c2}}{\bar{z}_{c2}})^2}\left(\frac{\partial \bar{x}_{c2}}{\partial f_x}\cdot\frac{1}{\bar{z}_{c2}} + \bar{x}_{c2}\cdot(-\frac{1}{\bar{z}_{c2}^2})\cdot\frac{\partial \bar{z}_{c2}}{\partial f_x}\right)$$

$$= \frac{C}{C^2+A^2}\left(\frac{\partial A}{\partial f_x} - \frac{A}{C}\cdot\frac{\partial C}{\partial f_x}\right)$$

$$\frac{\partial A}{\partial f_x} = P_2 P_1^{-1}\left(r_{11}\frac{\partial F(1)}{\partial f_x} + r_{12}\frac{\partial F(2)}{\partial f_x} + r_{13}\frac{\partial F(3)}{\partial f_x}\right)$$

$$= P_2 P_1^{-1}\big(r_{11}\cos lat_1 \cos lon_1 (-f_x^{-2})(u_1-c_x) + r_{12}\cdot 0 + r_{13}\cos lat_1 (-\sin lon_1)(-f_x^{-2})(u_1-c_x)\big)$$

$$= P_2 P_1^{-1} f_x^{-2}(u_1-c_x)\cos lat_1 (-r_{11}\cos lon_1 + r_{13}\sin lon_1)$$

$$\frac{\partial C}{\partial f_x} = P_2 P_1^{-1} f_x^{-2}(u_1-c_x)\cos lat_1 (-r_{31}\cos lon_1 + r_{33}\sin lon_1)$$

$$\frac{\partial lon_2}{\partial f_x} = \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2+\bar{x}_{c2}^2}\cdot P_2 P_1^{-1} f_x^{-2}(u_1-c_x)\cos lat_1\left[\cos lon_1\left(\frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{31} - r_{11}\right) + \sin lon_1\left(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{33}\right)\right]$$

$$\frac{\partial u_2}{\partial f_x} = lon_2 + f_x\frac{\partial lon_2}{\partial f_x}$$

$$= lon_2 + \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2+\bar{x}_{c2}^2}\cdot P_2 P_1^{-1} \underbrace{f_x^{-1}(u_1-c_x)}_{lon_1}\cos lat_1\left[\cos lon_1\left(\frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{31} - r_{11}\right) + \sin lon_1\left(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{33}\right)\right]$$

$$\frac{\partial lon_2}{\partial c_x} = \frac{C}{C^2+A^2}\left(\frac{\partial A}{\partial c_x} - \frac{A}{C}\cdot\frac{\partial C}{\partial c_x}\right)$$

$$\frac{\partial A}{\partial c_x} = P_2 P_1^{-1} f_x^{-1}\cos lat_1 (r_{13}\sin lon_1 - r_{11}\cos lon_1)$$

$$\frac{\partial C}{\partial c_x} = P_2 P_1^{-1} f_x^{-1}\cos lat_1 (r_{33}\sin lon_1 - r_{31}\cos lon_1)$$

$$\frac{\partial lon_2}{\partial f_{cx}} = \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2+\bar{x}_{c2}^2} P_2 P_1^{-1} f_x^{-1}\cos lat_1\left[\cos lon_1\left(\frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{31} - r_{11}\right) + \sin lon_1\left(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{33}\right)\right]$$

$$\frac{\partial u_2}{\partial c_x} = f_x\frac{\partial lon_2}{\partial c_x} + 1 = \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2+\bar{x}_{c2}^2} P_2 P_1^{-1}\cos lat_1\left[\cos lon_1\left(\frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{31} - r_{11}\right) + \sin lon_1\left(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}}r_{33}\right)\right] + 1$$

$$\frac{\partial lon_2}{\partial f_y} = \frac{C}{C^2+A^2}\left(\frac{\partial A}{\partial f_y} - \frac{A}{C}\cdot\frac{\partial C}{\partial f_y}\right)$$

$$\frac{\partial A}{\partial f_y} = \rho_2\rho_1^{-1}\left(r_{11}\frac{\partial F(1)}{\partial f_y} + r_{12}\frac{\partial F(2)}{\partial f_y} + r_{13}\frac{\partial F(3)}{\partial f_y}\right)$$

$$= \rho_2\rho_1^{-1} f_y^{-2}(v_1 - c_y)(r_{11}\sin lat_1 \sin lon_1 + r_{12}\cos lat_1 + r_{13}\sin lat_1 \cos lon_1)$$

$$\frac{\partial C}{\partial f_y} = \rho_2\rho_1^{-1} f_y^{-2}(v_1 - c_y)(r_{31}\sin lat_1 \sin lon_1 + r_{32}\cos lat_1 + r_{33}\sin lat_1 \cos lon_1)$$

$$\frac{\partial lon_2}{\partial f_y} = \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2}\rho_2\rho_1^{-1} f_y^{-2}(v_1 - c_y)\Big[\sin lat_1 \sin lon_1 \big(r_{11} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{31}\big) + \cos lat_1 \big(r_{12} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{32}\big) + \sin lat_1 \cos lon_1 \big(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{33}\big)\Big]$$

$$\frac{\partial u_2}{\partial f_y} = f_x \frac{\partial lon_2}{\partial f_y} =$$

$$= f_x f_y^{-1}\frac{\bar{z}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2}\rho_2\rho_1^{-1} f_y^{-1}(v_1 - c_y)\Big[\sin lat_1 \sin lon_1 \big(r_{11} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{31}\big) + \cos lat_1 \big(r_{12} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{32}\big) + \sin lat_1 \cos lon_1 \big(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{33}\big)\Big]$$

$$\frac{\partial lon_2}{\partial c_y} = \frac{C}{C^2+A^2}\left(\frac{\partial A}{\partial c_y} - \frac{A}{C}\cdot\frac{\partial C}{\partial c_y}\right)$$

$$\frac{\partial A}{\partial c_y} = \rho_2\rho_1^{-1} f_y^{-1}(r_{11}\sin lat_1 \sin lon_1 + r_{12}\cos lat_1 + r_{13}\sin lat_1 \cos lon_1)$$

$$\frac{\partial C}{\partial f_y} = \rho_2\rho_1^{-1} f_y^{-1}(r_{31}\sin lat_1 \sin lon_1 + r_{32}\cos lat_1 + r_{33}\sin lat_1 \cos lon_1)$$

$$\frac{\partial lon_2}{\partial c_y} = \frac{\bar{z}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2}\rho_2\rho_1^{-1} f_y^{-1}\Big[\sin lat_1 \sin lon_1 \big(r_{11} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{31}\big) + \cos lat_1 \big(r_{12} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{32}\big) + \sin lat_1 \cos lon_1 \big(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{33}\big)\Big]$$

$$\frac{\partial u_2}{\partial c_y} = f_x f_y^{-1}\frac{\bar{z}_{c2}}{\bar{z}_{c2}^2 + \bar{x}_{c2}^2}\rho_2\rho_1^{-1}\Big[\sin lat_1 \sin lon_1 \big(r_{11} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{31}\big) + \cos lat_1 \big(r_{12} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{32}\big) + \sin lat_1 \cos lon_1 \big(r_{13} - \frac{\bar{x}_{c2}}{\bar{z}_{c2}} r_{33}\big)\Big]$$

$$\frac{\partial lat_2}{\partial f_x} = -\frac{\partial \arcsin \bar{Y}_{c2}}{\partial f_x} = -\frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \frac{\partial \bar{Y}_{c2}}{\partial f_x}$$

$$\frac{\partial \bar{Y}_{c2}}{\partial f_x} = \frac{\partial P_2 P_1^{-1}(r_{21}F(1) + r_{22}F(2) + r_{23}F(3))}{\partial f_x}$$

$$= P_2 P_1^{-1}\left(r_{21}\frac{\partial F(1)}{\partial f_x} + r_{22}\frac{\partial F(2)}{\partial f_x} + r_{23}\frac{\partial F(3)}{\partial f_x}\right)$$

$$= P_2 P_1^{-1}(r_{21}\cos lat_1 \cos lon_1 (-f_x^{-2})(u_1 - c_x) + r_{22}\cdot 0 +$$
$$r_{23}\cos lat_1 (-\sin lon_1)(-f_x^{-2})(u_1-c_x))$$

$$= P_2 P_1^{-1} f_x^{-2}(u_1-c_x)\cos lat_1 (-r_{21}\cos lon_1 + r_{23}\sin lon_1)$$

- $$\frac{\partial v_2}{\partial f_x} = f_y \frac{\partial lat_2}{\partial f_x}$$

$$= \frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} f_y f_x^{-1} P_2 P_1^{-1} \underbrace{f_x^{-1}(u_1-c_x)}_{lon_1}\cos lat_1 (r_{21}\cos lon_1 - r_{23}\sin lon_1)$$

$$= -\frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} \qquad (r_{23}\sin lon_1 - r_{21}\cos lon_1)$$

$$\frac{\partial lat_2}{\partial c_x} = -\frac{\partial \arcsin \bar{Y}_{c2}}{\partial c_x} = -\frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \frac{\partial \bar{Y}_{c2}}{\partial c_x}$$

$$\frac{\partial \bar{Y}_{c2}}{\partial c_x} = P_2 P_1^{-1} f_x^{-1} \cos lat_1 (r_{23}\sin lon_1 - r_{21}\cos lon_1)$$

- $$\frac{\partial v_2}{\partial c_x} = f_y \frac{\partial lat_2}{\partial c_x} = \frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} f_y f_x^{-1} P_2 P_1^{-1}\cos lat_1 (r_{21}\cos lon_1 - r_{23}\sin lon_1)$$

$$\frac{\partial lat_2}{\partial f_y} = \frac{\partial \arcsin \bar{Y}_{c2}}{\partial f_y} = -\frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \frac{\partial \bar{Y}_{c2}}{\partial f_y}$$

$$\frac{\partial \bar{Y}_{c2}}{\partial f_y} = P_2 P_1^{-1} \left(r_{21} \frac{\partial F(1)}{\partial f_y} + r_{22} \frac{\partial F(2)}{\partial f_y} + r_{23} \frac{\partial F(3)}{\partial f_y} \right)$$

$$= P_2 P_1^{-1} f_y^{-2} (v_1 - c_y)(r_{21} \sin lat_1 \sin lon_1 + r_{22} \cos lat_1 + r_{23} \sin lat_1 \cos lon_1)$$

- $$\frac{\partial v_2}{\partial f_y} = lat_2 + f_y \frac{\partial lat_2}{\partial f_y}$$

$$= lat_2 - \frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} P_2 P_1^{-1} \underbrace{f_y^{-1} (v_1 - c_y)}_{lat_1} (r_{21} \sin lat_1 \sin lon_1 + r_{22} \cos lat_1 + r_{23} \sin lat_1 \cos lon_1)$$

$$\frac{\partial lat_2}{\partial c_y} = \frac{\partial \arcsin \bar{Y}_{c2}}{\partial c_y} = -\frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} \cdot \frac{\partial \bar{Y}_{c2}}{\partial c_y}$$

$$\frac{\partial \bar{Y}_{c2}}{\partial c_y} = P_2 P_1^{-1} f_y^{-1} (r_{21} \sin lat_1 \sin lon_1 + r_{22} \cos lat_1 + r_{23} \sin lat_1 \cos lon_1)$$

- $$\frac{\partial v_2}{\partial c_y} = f_y \frac{\partial lat_2}{\partial c_y} + 1$$

$$= 1 - \frac{1}{\sqrt{1-\bar{Y}_{c2}^2}} P_2 P_1^{-1} (r_{21} \sin lat_1 \sin lon_1 + r_{22} \cos lat_1 + r_{23} \sin lat_1 \cos lon_1)$$